*Int. J. Multiphase Flow* Vol. 7, No. 5, pp. 479–492, 1981
Printed in Great Britain.

0301-9322/81/050479–14$02.00/0
Pergamon/Elsevier

# THE INFLUENCE OF ROTATION IN SLOW VISCOUS FLOWS

S. H. SMITH
Department of Mathematics, University of Toronto, Toronto, Canada M5S 1A1

(*Received* 14 *July* 1980; *in revised form* 17 *November* 1980)

**Abstract**—We investigate the secondary streaming motion due to the slow rotation of an axisymmetric body in a rotating viscous fluid. We find in general that for all such bodies the velocities tend to zero at a distance $0(E^{1/2})$ from the body, where $E$ is the (large) Ekman number. Four specific geometries are considered: sphere, spheroid, spherical cap and the double sphere; in all except the first case a small Rossby number has assumed. The resultant translational force on the spherical cap, where there is no fore–aft symmetry, has been calculated. Further, separated flow in the last two cases can be displayed.

## 1. INTRODUCTION

When an axially symmetric body rotates about its axis of symmetry in a fluid with kinematic viscosity $\nu$, an angular velocity is imparted to the fluid through the action of viscous forces. A basic physical parameter for such flows is the Ekman number $E = \nu/\Omega a^2$, where $\Omega$ is a typical angular velocity and $a$ is a representative lateral dimension for the body. When $E$ is suitably large the resultant slow viscous motion is essentially a Stokes flow, and the angular velocity can be found to the first order in $E^{-1}$ by solving a simple second order partial differential equation. A number of solutions representing different geometries have been presented in the literature through the years.

It is well known that this angular velocity within the fluid develops a pressure gradient which, in its turn, must lead to a streaming motion in the fluid. However, there have been few studies of this slow viscous streaming flow, and these only for the case when the rotating body is a sphere. Childress (1964) extended the perturbation method of Proudman and Pearson in considering the extended problem of a uniform flow with velocity $U$ past a sphere which is also rotating. His main result was to calculate the correction to the Stokes drag formula valid for small Reynolds number $R$ (defined by $R = Ua/\nu$) and large Ekman number $E$ such that the product $R^2E$ is finite. Also, Ranger (1971) showed that the uniform flow past a rotating sphere can lead to separation. The main purpose of the present paper is to consider the secondary streaming motion alone for four different geometries in an attempt to develop an understanding of basic behaviours present.

It is seen that the magnitude of the velocities in this secondary streaming flow is $\Omega^2a^3/\nu$, and so its Reynolds number is small as $E^{-2}$. When the fluid at infinity is itself rotating (with a different angular velocity than that of the body) a further parameter, the Rossby number, must be introduced. If the angular velocity of the fluid is represented by $\Omega$, and of the body by $\Omega_1$, then an effective definition for the Rossby number is $|\Omega_1 - \Omega|/|\Omega_1 + \Omega|$; this is equivalent to half the classical Rossby number as defined for small Ekman number situations.

There is one general conclusion that can be reported straightaway. When the streaming flow is considered as a slow viscous flow with the forcing term due to the pressure gradient, then the Coriolis term in the Navier–Stokes equations is neglected; mathematically, this requires $E^2 \gg 1$. However, the resulting solution shows finite velocities at a large distance from the body. Consequently, the solution must be considered as an inner solution, and it is necessary to re-introduce the Coriolis term to give the corresponding outer solution. The distance scale is $0(E^{1/2})$, and the streaming velocities are indeed reduced to zero at infinity (see Herron *et al.* 1975).

The four geometries we consider here are (a) sphere, (b) spheroid, (c) spherical cap, (d) two equal non-intersecting spheres. The equation for the streaming flow is linear in the stream function, but the forcing term is quadratic in the angular velocity. Unfortunately, a solution to this equation has only been found when the body is a sphere if the quadratic term is retained. In the other three cases it has been necessary to restrict ourselves to situations where the angular velocity difference between that of the body and the fluid at infinity is small—i.e. where the Rossby number is small. However, in the first case with the sphere it is seen that there is no real qualitative difference between the finite and small Rossby number situations, and so there can be some confidence that the more limited solutions in cases (b)–(d) do give worthwhile information for the finite Rossby number situations.

The methods used in the present paper can be considered as a further application of the theory of Weinstein involving generalized axisymmetric potentials, and, in particular, follows as an extension of the work of Payne & Pell (1960) for standard Stokes flows past axisymmetric bodies. The analysis is lengthy in most of the problems considered, but once the basic extension has been presented (section 3), the details are often similar to those already in the literature, and so the results alone are given when appropriate.

In all cases, far from the body the basic secondary streaming motion is equivalent to that of a point source of angular momentum, and so it can be described (roughly) as an inward flow towards the body in the equatorial region with a compensating outward flow in the polar region. Closer to the body, there are only certain quantitative adjustments to the flow field for the case of the spheroid due to the presence of the solid surface; the qualitative pattern of the streamlines is similar to that for the angular momentum source. In cases (c) and (d) the local flow is qualitatively different because of the body geometry, and further examples of flow separation are found to add to those reviewed recently by O'Neill & Ranger (1979), and by Hasimoto & Sano (1980). For the spherical cap there is separated flow for a restricted range only of cap angles; the separated region is essentially contained within the cap. With two spheres, the separated region, which joins both the spheres, is present for all cases.

## 2. ROTATING SPHERE

It is perhaps best to present the case which leads to the most straightforward analysis before indicating the general theory. We introduce spherical polar co-ordinates $(r, \theta, \omega)$; $ar$ represents distance in the radial direction, with the surface of the sphere given by $r = 1$. The velocities in the three directions $\hat{\mathbf{r}}$, $\boldsymbol{\theta}$, $\hat{\boldsymbol{\omega}}$ can be written

$$\frac{\Omega a}{r^2 \sin\theta}\Psi_\theta, -\frac{\Omega a}{r\sin\theta}\Psi_r, \frac{\Omega a}{r\sin\theta}X \tag{2.1}$$

respectively, $\Psi(r, \theta)$ is the non-dimensional stream function and $X(r, \theta)$ the non-dimensional azimuthal velocity. With a slow viscous flow, the equation for $X$ is just

$$L_{-1}X = 0\,, \tag{2.2}$$

where $L_{-1}$ is the operator defined by

$$L_{-1} \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} - \frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta}.$$

The solution for $X$, which ensures the angular velocity equals $\lambda$ on $r = 1$ and tends to unity as $r \to \infty$, showing the Rossby number $|\lambda - 1|/|\lambda + 1|$, is

$$X = \{r^2 + (\lambda - 1)r^{-1}\}\sin^2\theta\,. \tag{2.3}$$

This angular velocity produces a pressure gradient that gives rise to the secondary streaming motion with stream function $\Psi$, where $\Psi$ satisfies the equation

$$EL_{-1}^2\Psi = \frac{2X}{r^2 \sin^2\theta}\left(X_r \cos\theta - \frac{1}{r}X_\theta \sin\theta\right). \qquad [2.4]$$

The r.h.s. is derived from the centrifugal acceleration term which is the driving force for the secondary streaming motion; all other non-linear terms in the Navier–Stokes equations have been neglected, as they contribute only to third and higher order terms for finite $r$ when $E^{-1}$ is small. The solution for $\Psi$ which satisfies the no-slip conditions on the sphere, and has the slowest growth at infinity, is

$$\Psi = -\frac{\lambda-1}{8E}\left\{2r^2 - (\lambda+3) + \frac{2(\lambda-1)}{r} - \frac{(\lambda-3)}{r^2}\right\}\cos\theta \sin^2\theta. \qquad [2.5]$$

The first point to note here is that the streaming velocity does not tend to zero for large $r$; when the fluid is at rest at infinity Ranger (1971) found no term in $r^2$. Secondly, when we put $\lambda = 1+\epsilon$, with $\epsilon \ll 1$ (i.e. the Rossby number is small, with the sphere having only a slightly different angular velocity from that of the fluid at infinity), then

$$\Psi \equiv \epsilon\psi = -\frac{1}{4}\epsilon E^{-1}(r^2 - 2 + r^{-2})\cos\theta\sin^2\theta\,; \qquad [2.6]$$

the error is $O(\epsilon^2)$. This expression could have been gained directly from the linearised version of [2.4], where $X$ is replaced by $r^2\sin^2\theta + \epsilon\chi$, and the quadratic terms in $\epsilon$ neglected to show

$$EL_{-1}^2\psi = 2(\chi_r\cos\theta - r^{-1}\chi_\theta\sin\theta). \qquad [2.7]$$

It is noted that the qualitative difference between the flows described by [2.5] and [2.6] is indeed slight, and gives a definite indication that the Rossby number has a limited role in the type of behaviour being considered here.

Returning to the first point, finite streaming velocities are not physically possible an infinite distance from the sphere, and must be a consequence of the restricted equation [2.4] (or [2.6] in the linear case); terms have been ignored in the Navier–Stokes equations that are important a long distance from the sphere. In effect, it is necessary that an "Oseen" type approximation, based on uniform angular velocity at infinity, be taken to bring the velocities at infinity to zero.

To this end, although for simplicity we restrict the arguments following to the linear situation with $\epsilon \ll 1$, we introduce cylindrical co-ordinates where $a\rho$ and $az$ represent lengths in the radial and axial directions. (In the remainder of this section the term "radial" is used in the cylindrical sense.) The velocities in the radial, azimuthal and axial directions can be written as

$$-\frac{\epsilon\Omega a}{\rho}\psi_z,\ \Omega a\left(\rho + \frac{\epsilon\chi}{\rho}\right),\ \frac{\epsilon\Omega a}{\rho}\psi_\rho \qquad [2.8]$$

so that the linearized equations become

$$L_{-1}\chi = 0,\ 2\chi_z = EL_{-1}^2\psi \qquad [2.9]$$

corresponding to [2.2] and [2.7] respectively; in cylindrical co-ordinates

$$L_{-1} \equiv \frac{\partial^2}{\partial\rho^2} - \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{\partial^2}{\partial z^2}.$$

However, if we make the stretching transformations for large $\rho$, $z$ in the Navier–Stokes equations, with

$$\bar{\rho} = E^{-1/2}\rho, \bar{z} = E^{-1/2}z, \bar{\psi} = \psi, \bar{\chi} = E^{1/2}\chi,$$

then there results the "Oseen" equations

$$2\bar{\chi}_{\bar{z}} = \bar{L}^2_{-1}\bar{\psi}, -2\bar{\psi}_{\bar{z}} = \bar{L}_{-1}\bar{\chi} \qquad [2.10]$$

where $\bar{L}_{-1}$ is the $L_{-1}$ operator in terms of $\bar{\rho}$ and $\bar{z}$. The Coriolis term has now been re-introduced into the governing equations. We now consider the solution of [2.10] that shows the velocities tend to zero as $\bar{r} = (\bar{\rho}^2 + \bar{z}^2)^{1/2} \to \infty$, and represents the motion due to a rotating sphere as $\bar{r} \to 0$. Because the required solution will be an outer solution in the sense of Proudman and Pearson, it is sufficient to consider a point source of rotation at $\bar{r} = 0$ which will represent all finite rotating bodies (to within some multiplicative constant). Consequently, it is sufficient to find the solution of

$$2\bar{\chi}_{\bar{z}} = \bar{L}^2_{-1}\bar{\psi} \text{ and } -2\bar{\psi}_{\bar{z}} - 2\bar{\rho}^{-1}\delta(\bar{\rho})\delta(\bar{z}) = \bar{L}_{-1}\bar{\chi} \qquad [2.11]$$

that tends to zero at infinity.

This pair of equations can be solved by taking Hankel transforms in $\bar{\rho}$ and Fourier transforms in $\bar{z}$. The details are straightforward and show

$$\bar{\chi} = \bar{\rho}\int_0^\infty k^2\left[\frac{(k^2-\alpha_1{}^2)^2\,\mathrm{e}^{-\alpha_1\bar{z}}}{\alpha_1(\alpha_2{}^2-\alpha_1{}^2)(\alpha_3{}^2-\alpha_1{}^2)} + \frac{(k^2-\alpha_2{}^2)\,\mathrm{e}^{-\alpha_2\bar{z}}}{\alpha_2(\alpha_1{}^2-\alpha_2{}^2)(\alpha_3{}^2-\alpha_2{}^2)} + \frac{(k^2-\alpha_3{}^2)^2\,\mathrm{e}^{-\alpha_3\bar{z}}}{\alpha_3(\alpha_1{}^2-\alpha_3{}^2)(\alpha_2{}^2-\alpha_3{}^2)}\right]J_1(k\bar{\rho})\,\mathrm{d}k \qquad [2.12]$$

where $\alpha_1$, $\alpha_2$, $\alpha_3$ are the roots of $(\alpha^2-k^2)^3+4\alpha^2=0$ with positive real parts. A similar expression follows for $\bar{\psi}$. To evaluate the integral [2.12] for small $\bar{r}$, it is sufficient to approximate the three roots for $\alpha$ by $k-(2k)^{-1/3}$, $k+\mathrm{e}^{i\pi/3}(2k)^{-1/3}$ and $k+\mathrm{e}^{-i\pi/3}(2k)^{-1/3}$, and then the square bracket gives just the contribution $k^{-1}\,\mathrm{e}^{-k\bar{z}}$, from which it can be confirmed that

$$\chi \simeq r\int_0^\infty \kappa\,\mathrm{e}^{-\kappa z}J_1(\kappa r)\,\mathrm{d}\kappa = r^{-1}\sin^2\theta$$

as $\bar{r} \to 0$ (see [2.3]). Correspondingly, $\psi \sim -(1/4)E^{-1}r^2\cos\theta\sin^2\theta$ as $\bar{r} \to 0$. Together, these represent the azimuthal velocity and secondary streaming motion for a point source of angular momentum at the origin in a rotating fluid. When $\bar{r}$ is large, the roots for $\alpha$ can be taken as aprox. $(1/2)k^3$ and $1 \pm i$; only the first contributes to the dominant term which is seen to lead to

$$\bar{\psi} \sim -\frac{\bar{\rho}}{2\bar{r}^3}\int_0^\infty u^2\exp\left(-\frac{u^3\bar{z}}{2\bar{r}^3}\right)J_1\left(\frac{u\bar{\rho}}{\bar{r}}\right)\mathrm{d}u.$$

This integral has differing asymptotic approximations depending on the value of $\mu = \bar{z}/\bar{r}^3$. When $\mu \gg 1$, the integral has magnitude $O(\bar{\rho}^{-2}\mu^{-4/3})$, and when $\mu \ll 1$, it is small as $\exp(-2/3 \cdot 3^{-1/2}\mu^{-1/2})$. Hence the streaming velocities are algebraically small close to the axis of rotation as $\bar{r} \to \infty$, and exponentially small far from the axis.

This is the desired result; the Stokes theory (where the Coriolis term is neglected) has now been properly embedded in an Oseen theory. In each of the other examples that follow, it can be seen that the only alteration in the Oseen theory will be the introduction of a multiplicative factor for the delta function in [2.11] to represent the size of the sphere to which the body is equivalent at a large distance.

## 3. GENERAL THEORY FOR A FINITE ROTATING BODY

In the present section we briefly indicate the necessary extension to the general approach of Payne & Pell (1960) when rotating flows with low Rossby numbers are being considered. The main results they used were derived from the generalized axisymmetric potential theory that had been developed earlier by Weinstein. When $\psi^{(k)}(\rho, z)$ denotes any solution of

$$L_k(v) \equiv v_{\rho\rho} + k\rho^{-1}v_\rho + v_{zz} = 0\,, \tag{3.1}$$

in some suitably simple domain $D$, then the following expressions are solutions of the basic Stokes flow equation $L_{-1}^2(v) = 0$ for the stream function:

$$\text{(a)}\ \ \rho^2\psi^{(3)},\ \text{(b)}\ \ z\rho^2\psi^{(3)},\ \text{(c)}\ \ r^2\rho^2\psi^{(3)},\ \text{(d)}\ \ \rho^2\psi^{(1)},\ \text{(e)}\ \ \rho^4\psi^{(5)}. \tag{3.2}$$

Further, any solution of the Stokes equation can be represented as a linear combination of any two of the expressions [3.2].

Now the equation to be solved to represent the secondary flow is

$$EL_{-1}^2\psi = 2\chi_z\,, \tag{3.3}$$

where

$$L_{-1}\chi = 0\,. \tag{3.4}$$

Eliminating $\chi$ between [3.3] and [3.4] shows that $L_{-1}^3\psi = 0$. The extra result we need is a method for finding the particular integral of [3.3]: the rest then follows from the earlier theory. In fact, when we write $\psi = z\omega$, where $L_{-1}^2\omega = 0$, then certainly $L_{-1}^3\psi = 0$. But $L_{-1}^2\psi = zL_{-1}^2\omega + 4L_{-1}\omega_z$, and so from [3.3] it is seen that

$$L_{-1}\omega = \frac{1}{2}E^{-1}\chi\,. \tag{3.5}$$

In particular, from [3.5], we can see that

$$\text{(a)}\ \ \omega = \rho^2\psi^{(1)} \quad \text{implies} \quad 4\rho\psi_\rho^{(1)} = E^{-1}\chi\,, \tag{3.6}$$

$$\text{(b)}\ \ \omega = \rho^4\psi^{(5)} \quad \text{implies} \quad 4\rho^3\psi_\rho^{(5)} + 16\rho^2\psi^{(5)} = E^{-1}\chi\,, \tag{3.7}$$

$$\text{(c)}\ \ \omega = r^2\rho^2\psi^{(3)} \quad \text{implies} \quad 8\rho^3\psi_\rho^{(3)} + 20\rho^2\psi^{(3)} + 8z\rho^2\psi_\rho^{(3)} = E^{-1}\chi\,; \tag{3.8}$$

we use these results in turn, and when appropriate, in the remaining sections of this paper.

## 4. OBLATE SPHEROID

We introduce elliptic co-ordinates through

$$z = \sinh\xi\cos\eta \quad \rho = \cosh\xi\sin\eta\,,$$

and the azimuthal velocity $\chi$ is given by

$$\chi_{\xi\xi} + \chi_{\eta\eta} - \coth\xi\chi_\xi - \cot\eta\cdot\chi_\eta = 0 \tag{4.1}$$

from [3.4]. The solution that satisfies the condition $\chi = 1$ on the spheroid $\xi = \xi_0(>0)$, and tends to zero a large distance from the body is

$$\chi = A \sin^2 \eta\{(\tau^2+1)\operatorname{arc\,cot}\tau - \tau\} \qquad [4.2]$$

where $\tau = \sinh \xi$, and $A = (\tau_0{}^2+1)\{(\tau_0{}^2+1)\operatorname{arc\,cot}\tau_0 - \tau_0\}^{-1}$. The radius of the sphere to which this rotating spheroid is equivalent as $r \to \infty$ is $2/3(\tau_0{}^2+1)^{-3/2}Aa$; which is $4a/3\pi$ for the limiting case of a finite disc of radius $a$ (where $\tau_0 = 0$).

We next write

$$\psi = z\rho^2(\psi^{(1)} + \psi^{(3)} + \rho^2\psi^{(5)})$$

for the stream function. The calculation of $\psi$ is very similar to that developed by Payne & Pell (1960), so it is sufficient to report the solution

$$\begin{aligned}\psi = \frac{A}{8E}(\tau^2+1)\cos\eta\sin^2\eta[\{&-(2\tau^2+2-3\tau^2\sin^2\eta-\sin^2\eta)-\alpha(12\tau^2+4-3\sin^2\eta\\&-15\tau^2\sin^2\eta)-\beta+3\gamma(5\tau^2+1)(4-5\sin^2\eta)+3\delta\}\tau\operatorname{arc\,cot}\tau\\&+\{\beta(\tau^2+1)^{-1}-\gamma(9\tau^2+7)(\tau^2+1)^{-1}(4-5\sin^2\eta)+3\alpha(4-5\sin^2\eta)\\&+2\alpha(\tau^2+1)^{-1}\sin^2\eta+(2-3\sin^2\eta)\}]\,;\end{aligned} \qquad [4.3]$$

$\alpha$, $\beta$, $\delta$ and $\delta$ are constants that are immediately determined by satisfying the conditions $\psi = \psi_\tau = 0$ on $\tau = \tau_0$.

In the particular case with a flat disc of radius $a$, these constants have been evaluated to show

$$\psi = \frac{\tau^2}{20E}\cos\eta\sin^2\eta[12\tau(\tau^2+1)\operatorname{arc\,cot}\tau\cdot\cos^2\eta - 4(3\tau^2+2)\cos^2\eta - 5]\,. \qquad [4.4]$$

The corresponding result for a prolate spheroid follows in a similar fashion.

## 5. SPHERICAL CAP

Because of its role in the development of many slow viscous flow problems, we consider this geometry in a little more detail. The method of solution follows fairly closely that developed by Dorrepaal *et al.* (1976) for the uniform flow past a spherical cap.

Spherical polar co-ordinates are utilised, with the cap given by $r = 1$, $0 \le \theta \le \alpha$. We introduce the integral representations (for different values of $j$)

$$\begin{aligned}V_j(r,\theta) &= r^{1/2}\int_0^\theta \frac{v_j(r,\lambda)\sin\lambda\cdot d\lambda}{(\cos\lambda-\cos\theta)^{1/2}} = -r^{1/2}\int_\theta^\pi \frac{u_j(r,\lambda)\sin\lambda\cdot d\lambda}{(\cos\theta-\cos\lambda)^{1/2}},\ r<1\,,\\ &= r^{1/2}\int_0^\theta \frac{v_j(r^{-1},\lambda)\sin\lambda\cdot d\lambda}{(\cos\lambda-\cos\theta)^{1/2}} = -r^{1/2}\int_\theta^\pi \frac{u_j(r^{-1},\lambda)\sin\lambda\cdot d\lambda}{(\cos\lambda-\cos\theta)^{1/2}},\ r>1\,.\end{aligned} \qquad [5.1]$$

These representations were first introduced by Ranger (1972) for the solution of problems involving the spherical cap geometry. The functions $V_j(r,\theta)$ satisfy $L_{-1}(V_j) = 0$ when $u_j$ and $v_j$ are conjugate two dimensional harmonics, which in turn can be expressed in the form

$$u_j + iv_j = \frac{2\sqrt{2}}{\pi}\sum_{n=1}^{\infty} A_n{}^{(j)}(r\,e^{i\lambda})^{n+(1/2)} \qquad [5.2]$$

for real coefficients $A_n^{(j)}$. Ranger showed that $V_j$ can also be given as the series

$$V_j(r,\theta)=\sum_{n=1}^{\infty}A_n^{(j)}r^{n+1}\{P_{n-1}(\cos\theta)-P_{n+1}(\cos\theta)\},\ r<1\,,$$

$$=\sum_{n=1}^{\infty}A_n^{(j)}r^{-n}\{P_{n-1}(\cos\theta)-P_{n+1}(\cos\theta)\},\ r>1\,, \qquad [5.3]$$

where $P_n(\cos\theta)$ are Legendre polynomials. The value of these representations [5.1], as demonstrated by Dorrepaal *et al.*, is that $v_j(1,\lambda)$ can be found upon satisfying the boundary conditions on $r=1$; from [5.2] the coefficients $A_n^{(j)}$ are then determined to give $V_j(r,\theta)$ in series form through [5.3]. For the azimuthal velocity we now write $\chi=V_0(r,\theta)$. It follows that $V_0=\sin^2\theta$ on $r=1$ for $0\le\theta\le\alpha$ to satisfy the no-slip condition on the cap, and solving the resulting integral equation

$$\int_0^{\theta}\frac{v_0(1,\lambda)\sin\lambda\,\mathrm{d}\lambda}{(\cos\lambda-\cos\theta)^{1/2}}=\sin^2\theta,\quad 0\le\theta\le\alpha\,,$$

gained from [5.1] with $j=0$, shows

$$v_0(1,\lambda)=\frac{4\sqrt{2}}{3\pi}\sin\frac{3}{2}\lambda,\quad 0\le\lambda\le\alpha\,; \qquad [5.4]$$

the same equation was solved by Dorrepaal *et al.*

Further, $V_{0r}$ is continuous along $r=1$ when $\alpha\le\theta\le\pi$, which requires $u_{0r}(1,\lambda)=0$, whence $v_0(1,\lambda)=$ constant for $\alpha\le\lambda\le\pi$. Because $V_0(1,\pi)$ must be zero, the value of the constant can be evaluated to show

$$v_0(1,\lambda)=-\frac{4\sqrt{2}}{3\pi}\sin^3\frac{\alpha}{2},\quad \alpha\le\lambda\le\pi\,. \qquad [5.5]$$

The solution for $V_0(r,\theta)$ is completed by calculating

$$A_n^{(0)}=2^{-1/2}\int_0^{\pi}v_0(1,\lambda)\sin\left(n+\frac{1}{2}\right)\lambda\cdot\mathrm{d}\lambda$$

$$=\frac{2}{3\pi}\left\{\frac{\sin(n-1)\alpha}{n-1}-\frac{\sin(n+2)\alpha}{n+2}\right\}-\frac{8}{3\pi}\sin^3\frac{\alpha}{2}\,\frac{\cos\left(n+\frac{1}{2}\right)\alpha}{2n+1}\,. \qquad [5.6]$$

The radius of the equivalent sphere for the flow at infinity is $(3/2)A_1^{(0)}=a\pi^{-1}(\alpha-(1/2)\sin 2\alpha+(2/3)\sin^3\alpha)$; this result was previously given by Collins (1963) using dual series methods.

For the secondary streaming flow, we know from section 3 that $L_{-1}^3(r\cos\theta\cdot\omega)=0$ when $L_{-1}^2(\omega)=0$. Now a general representation for $\omega(r,\theta)$ which satisfies $L_{-1}^2\omega=0$ is

$$\omega=-V_1+(r^2-1)\left(\frac{1}{2}rV_{1r}-V_2\right), \qquad [5.7]$$

where $L_{-1}(V_j)=0$ (Ranger 1972). Therefore, we write $\psi=r\cos\theta\cdot\omega$, with $\omega$ given by [5.7], and it follows that the particular integral for the required equation [2.7] is given by

$$2(2r^2V_{1rr}+3rV_{1r})-4(V_2+2rV_{2r})=E^{-1}\chi, \qquad [5.8]$$

upon using [3.5].

Consequently,

$$\psi=-V_3+(r^2-1)\left(\frac{1}{2}rV_{3r}-V_4\right)+r\cos\theta\{-V_1+(r^2-1)\left(\frac{1}{2}rV_{1r}-V_2\right)\} \qquad [5.9]$$

is the appropriate general expression for $\psi(r,\theta)$, with $V_1$ and $V_2$ given jointly (in a manner soon to be discovered) by [5.8], and we now proceed to calculate the functions $V_j$, $j=1,2,3,4$.

First, using the series expansions [5.3] for $r<1$ and $r>1$ in turn, a pair of simultaneous equations are found for $A_n^{(1)}$ and $A_n^{(2)}$, with solutions

$$(2n-1)(2n+3)A_n^{(1)}=A_n^{(0)} \text{ and } (2n-1)(2n+3)A_n^{(2)}=\frac{3}{4}A_n^{(0)}. \qquad [5.10]$$

Consequently,

$$V_2(r,\theta)\equiv\frac{3}{4}V_1(r,\theta). \qquad [5.11]$$

When the values [5.10] are substituted into the series [5.2], it follows that $4(\cos\lambda\cdot v_1)_{\lambda\lambda}+8(\sin\lambda\cdot v_1)_\lambda=-\cos\lambda\cdot v_0$ when $r=1$ for all $\lambda$. With $v_0(1,\lambda)$ known from [5.4], [5.5], we can now calculate

$$\begin{aligned} v_1(1,\lambda)&=\frac{4\sqrt{2}}{15\pi}\sin\frac{3\lambda}{2}+A_1\sin\lambda+A_2\cos\lambda \quad 0\le\lambda\le\alpha,\\ &=\frac{\sqrt{2}}{3\pi}\sin^3\frac{\alpha}{2}+A_3\sin\lambda+A_4\cos\lambda \quad \alpha\le\lambda\le\pi, \end{aligned} \qquad [5.12]$$

where $A_n$ $(n=1,2,3,4)$ are arbitrary constants; these, together with $v_2(1,\lambda)=(3/4)v_1(1,\lambda)$, complete the solution for the particular integral.

The next step is to satisfy the condition $\psi=0$ on $r=1$ for $0\le\theta\le\alpha$, which implies $V_3(1,\theta)+\cos\theta V_1(1,\theta)=0$ for $0\le\theta\le\alpha$. When we use the integral representations for $V_j$ $(j=1,3)$, plus the identity

$$\cos\theta\int_0^\theta(\cos\lambda-\cos\theta)^{-1/2}v_1\sin\lambda\,d\lambda=\int_0^\theta(\cos\lambda-\cos\theta)^{1/2}\{2(v_1\cos\lambda)_\lambda-v_1\sin\lambda\}\,d\lambda,$$

it follows that

$$v_{3\lambda}=\frac{3}{2}\sin\lambda\cdot v_1-\cos\lambda\cdot v_{1\lambda} \text{ for } r=1, \quad \theta\le\lambda\le\alpha.$$

Integration leads to

$$v_3(1,\lambda)=-\frac{\sqrt{2}}{25\pi}\sin\frac{5\lambda}{2}-\frac{5}{8}A_2\cos2\lambda-\frac{1}{8}A_1(5\sin2\lambda-2\lambda)+A_5, \quad 0\le\lambda\le\alpha. \qquad [5.13]$$

Further, $\psi_r = 0$ on $r = 1$ for $0 \le \theta \le \alpha$, and arguments similar to those used in calculating $v_3$ shows

$$v_4(1,\lambda) = -\frac{\sqrt{2}}{20}\sin\frac{5\lambda}{2} - \frac{25}{32}A_2\cos 2\lambda - \frac{5}{32}A_1(5\sin 2\lambda - 2\lambda) + A_6, \quad 0 \le \lambda \le \alpha. \qquad [5.14]$$

Off the surface of the cap, it is necessary to satisfy the conditions that $\psi(r,\theta)$ and all its derivatives are continuous across $r = 1$ when $\alpha \le \theta \le \pi$. Now the representation chosen for $\psi$ automatically ensure that both $\psi$ and $\psi_r$ satisfy this criterion. From the conditions for $\psi_{rr}$ and $\psi_{rrr}$ it follows, following closely the somewhat extended arguments of Dorrepaal *et al.* (1976), that

$$v_3(1,\lambda) = A_7\sin\lambda + A_8\cos\lambda + A_9\lambda + A_{10}, \quad \alpha < \lambda \le \pi, \qquad [5.15]$$

$$v_4(1,\lambda) = \frac{3}{4}A_7\sin\lambda + \frac{3}{4}A_8\cos\lambda + \frac{3}{4}A_9\lambda + A_{11}, \quad \alpha < \lambda \le \pi. \qquad [5.16]$$

There are eleven remaining constants to be found before the solution is complete. An immediate reduction is possible, for satisfying $v_j = 0$ at $r = 1$, $\lambda = 0$, plus $v_{j\lambda} = v_{j\lambda\lambda\lambda} = 0$ at $r = 1$, $\lambda = \pi$ from [5.2], shows $A_2 = A_3 = A_5 = A_6 = A_7 = A_9 = 0$. Next, because the coefficients $A_n^{(j)}$ are given by the inverse of [5.2] (similar to [5.6]), and $A_n^{(1)}$ and $A_n^{(2)}$ are known from [5.10], the constants $A_1$ and $A_4$ can be calculated from [5.12]. It is found that

$$A_1 = -\frac{2\sqrt{2}}{5\pi}\cos^5\frac{\alpha}{2}, \; A_4 = \frac{\sqrt{2}}{5\pi}\left(5\sin^3\frac{\alpha}{2} - 2\sin^5\frac{\alpha}{2}\right). \qquad [5.17]$$

The three remaining constants are determined by satisfying $V_3(1,\pi) = V_4(1,\pi) = 0$ (after noting that $V_1(1,\pi)$ and $V_2(1,\pi)$ are already zero), and finally by showing that $\lim_{\theta\to\alpha}(V_{3\theta} + \cos\alpha V_{1\theta})_{r=1}$ is finite to ensure that the velocities are bounded at the rim $r = 1$, $\theta = \alpha$. These calculations require finding the expressions for $V_j(1,\theta)$ through the integrals [5.1]. The details still closely follow Dorrepaal *et al.* and lead to the results

$$A_8 = -\frac{\sqrt{2}}{5\pi}\sin^3\frac{\alpha}{2}\left(5 - 16\sin^2\frac{\alpha}{2} + 6\sin^4\frac{\alpha}{2}\right),$$

$$A_{10} = \frac{\sqrt{2}}{50\pi}\left\{2\sin\frac{\alpha}{2}\left(21 - 47\cos^2\frac{\alpha}{2} + 16\cos^4\frac{\alpha}{2} + 25\cos^6\frac{\alpha}{2} - 10\cos^8\frac{\alpha}{2}\right) - 5\alpha\cos^5\frac{\alpha}{2}\right\}$$

$$A_{11} = \frac{\sqrt{2}}{120\pi}\left\{2\sin\frac{\alpha}{2}\left(33 - 67\cos^2\frac{\alpha}{2} + 24\cos^4\frac{\alpha}{2} + 31\cos^6\frac{\alpha}{2} - 6\cos^8\frac{\alpha}{2}\right) - 15\alpha\cos^5\frac{\alpha}{2}\right\}. \qquad [5.18]$$

The coefficients $A_n^{(j)}$ can now be obtained directly to complete the formal solution.

Some general results of physical interest can be developed at this juncture. First, the lack of fore-aft symmetry with the spherical cap implies that the secondary streaming flow will lead to a force on the cap in the axial direction. The general result for the drag given by Payne and Pell leads here to the expression

$$F = 2\pi\epsilon\mu\Omega a^2\{3(A_1^{(3)} + 2A_1^{(4)}) + 2(A_2^{(1)} + A_2^{(2)})\}.$$

These coefficients can be calculated to show the resultant force

$$F = \frac{128}{5}\epsilon\mu\Omega a^2\sin^3\frac{\alpha}{2}\cdot\cos^5\frac{\alpha}{2}\left(1 + \cos^2\frac{\alpha}{2}\right), \qquad [5.19]$$

in the direction of $\theta = \pi$; this is an attractively simple formula. The maximum value for $F$ is $2.982\,\epsilon\mu\Omega a^2$ when $\alpha = \alpha_0 \simeq 71.5°$.

To investigate the possibility of a separated flow we consider the representation for $\psi$ in the neighbourhood of the rim. A system of local polar co-ordinates $(\zeta, \sigma)$ is defined by $r = 1 - \zeta \sin\sigma$, $\theta = \alpha + \zeta\cos\sigma$, centred on the rim. An extended analysis then shows that the leading term for small $\zeta$ is

$$\psi \simeq \frac{16\surd(2)\zeta^{3/2}}{15\pi}\surd(\sin\alpha)\sin^2\frac{\alpha}{2}\cos^3\frac{\alpha}{2}\cos^3\frac{\sigma}{2}\left\{\left(3-\cos^2\frac{\alpha}{2}-2\cos^4\frac{\alpha}{2}\right)\right.$$

$$\left.-\tan\frac{\alpha}{2}\tan\frac{\sigma}{2}\left(5-4\cos^2\frac{\alpha}{2}-6\cos^4\frac{\alpha}{2}\right)\right\}.$$

Therefore, the streamline $\psi = 0$ leaves the rim at an angle $\sigma = \sigma_0$, where

$$\tan\frac{\sigma_0}{2} = \cot\frac{\alpha}{2}\left(3-\cos^2\frac{\alpha}{2}-2\cos^4\frac{\alpha}{2}\right)\left(5-4\cos^2\frac{\alpha}{2}-6\cos^4\frac{\alpha}{2}\right)^{-1}.$$

Now separation occurs when the above formula leads to a positive angle, with the free streamline $\psi = 0$ being the dividing line in the fluid for the separated flow region, which is essentially inside the cap. This domain is found to be given by $\alpha_c \leq \alpha \leq \pi$, where $\alpha_c \simeq 73.9°$.

The separation streamline $\psi = 0$ is drawn in figure 1 for two representative cases $\alpha = \pi/2$ and $\alpha = 2\pi/3$.

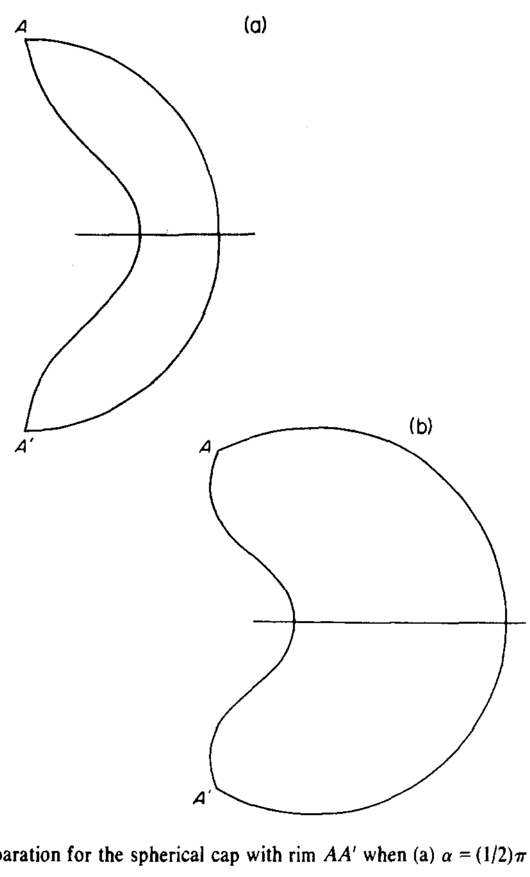


Figure 1. Regions of separation for the spherical cap with rim $AA'$ when (a) $\alpha = (1/2)\pi$ and (b) $\alpha = (2/3)\pi$.

## 6. TWO SPHERES

The Stokes flow past two spheres (the centres of both spheres lie on the axis of symmetry) was considered by Davis *et al.* (1976), and we follow their notation in this section. Bipolar co-ordinates $\xi$, $\eta$ are defined by

$$\rho = \frac{c \sin \eta}{\cosh \xi - \cos \eta}, \; z = \frac{c \sinh \xi}{\cosh \xi - \cos \eta} \; (c > 0) \; ; \tag{6.1}$$

the spheres $(z \pm c \coth \xi_0)^2 + \rho^2 = c^2 \operatorname{cosech}^2 \xi_0$ are represented by $\xi = \pm \xi_0$. We assume here that the spheres both rotate about the $z$-axis with angular velocity $\Omega(1+\epsilon)$, with the fluid at infinity having the angular velocity $\Omega$; there is a symmetry about $z = 0$. In the new co-ordinates the operator

$$L_{-1} \equiv \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} + \frac{\sinh \xi}{\cosh \xi - \cos \eta} \frac{\partial}{\partial \xi} + \frac{1 - \cosh \xi \cos \eta}{\cosh \xi - \cos \eta} \frac{\partial}{\partial \eta},$$

and the general solution of $L_{-1}\, \chi = 0$, which is even in $\xi$, can be written as

$$\chi = (\cosh \xi - \cos \eta)^{1/2} \sin^2 \eta \sum_{n=0}^{\infty} a_n \cosh \left(n + \frac{1}{2}\right) \xi \cdot P_n'(\mu), \tag{6.2}$$

where $\mu = \cos \eta$. When we require $\chi = \rho^2$ on $\xi = \pm \xi_0$, the coefficients $a_n$ are found to be

$$a_n = 2\surd(2) c^2 \, e^{-(n+(1/2))\xi_0} \operatorname{sech} \left(n + \frac{1}{2}\right) \xi_0, \; n = 0, 1, 2, \ldots. \tag{6.3}$$

It follows that the action of the two spheres at a large distance is equivalent to that of a single sphere with radius

$$\left\{ \surd(2c) \sum_{n=0}^{\infty} n(n+1) a_n \right\}^{1/3} a,$$

from which the torque to maintain the motion can readily be found.

To find the secondary streaming flow, we represent the particular integral for the solution of $EL_{-1}^2 \psi = 2\chi_z$ by

$$\psi_p = z\rho^2 \psi^{(1)} = \frac{c^2 \sinh \xi \sin^2 \eta}{E(\cosh \xi - \cos \eta)^{5/2}} \sum_{n=0}^{\infty} d_n \cosh \left(n + \frac{1}{2}\right) \xi \cdot P_n(\mu). \tag{6.4}$$

From [3.6], and after utilising many of the basic properties of Legendre polynomials, it can be shown that this leads to

$$2d_n - d_{n-1} - d_{n+1} = \frac{1}{2} a_n, \quad n = 1, 2, 3, \ldots. \tag{6.5}$$

When we define $e_n = d_n - d_{n-1}$ for $n = 1, 2, 3, \ldots$, the difference equation [6.5] becomes $e_n - e_{n+1} = (1/2) a_n$ for $n = 1, 2, 3, \ldots$. The unique solution for $e_n$, which is exponentially small as $n \to \infty$, is then given by

$$e_n = \frac{1}{2} \sum_{i=n}^{\infty} a_i, \quad n = 1, 2, 3, \ldots.$$

Hence, the unique solution for $d_n$, with the required behaviour that $d_n \to 0$ exponentially as $n \to \infty$, is

$$d_n = -\sum_{i=n+1}^{\infty} e_i, \quad n = 0, 1, 2, \ldots,$$

to complete the solution for the particular integral.

We now add to $\psi_p$ the general solution of $L_{-1}^2 \psi = 0$, that can be written as

$$\psi_c = \frac{c^2}{E(\cosh \xi - \cos \eta)^{3/2}} \sum_{n=1}^{\infty} \left\{ b_n \sinh \left(n - \frac{1}{2}\right) \xi + c_n \sinh \left(n + \frac{3}{2}\right) \xi \right\} \{P_{n-1}(\mu) - P_{n+1}(\mu)\}. \tag{6.6}$$

When [6.4] and [6.6] are added together, we find that the general solution for $\psi$ is most conveniently expressed as the single series

$$\begin{aligned}\psi = \frac{c^2}{E(\cosh \xi - \cos \eta)^{5/2}} \sum_{n=1}^{\infty} &\left\{ r_n \sinh \left(n - \frac{3}{2}\right) \xi + s_n \sinh \left(n + \frac{1}{2}\right) \xi \right. \\ &+ s_n \sinh \left(n + \frac{1}{2}\right) \xi \\ &\times \{P_{n-1}(\mu) - P_{n+1}(\mu)\}\end{aligned} \tag{6.7}$$

where

$$\begin{aligned} r_n &= -\frac{n(n+1)}{2(2n-1)(2n+1)} d_{n-1} + \frac{1}{2} b_n - \frac{n+1}{2n+1} b_{n-1}, \\ s_n &= \frac{n(n+1)}{2(2n+1)} \left\{ \frac{d_{n-1}}{2n-1} + \frac{d_{n+1}}{2n+3} \right\} + \frac{1}{2} b_n + \frac{1}{2} c_n - \frac{n}{2n+1} b_{n+1} - \frac{n+1}{2n+1} c_{n-1}, \\ t_n &= -\frac{n(n+1)}{2(n+1)(n+3)} d_{n+1} + \frac{1}{2} c_n - \frac{n}{2n+1} c_{n+1}, \end{aligned} \tag{6.8}$$

for $n = 1, 2, 3, \ldots$; for consistency, $b_0 = c_0 = 0$ in the above formulae. These difference relations [6.8] are coupled, and in effect give a single expression relating the three coefficients $r_n$, $s_n$, $t_n$; no simplification appears to be possible. The two final expressions between the coefficients follows from satisfying the no-slip conditions on $\xi = \pm \xi_0$, to give

$$r_n \sinh \left(n - \frac{3}{2}\right) \xi_0 + s_n \sinh \left(n + \frac{1}{2}\right) \xi_0 + t_n \sinh \left(n + \frac{5}{2}\right) \xi_0 = 0$$

$$(2n-3) r_n \cosh \left(n - \frac{3}{2}\right) \xi_0 + (2n+1) s_n \cosh \left(n + \frac{1}{2}\right) \xi_0 + (2n+5) t_n \cosh \left(n + \frac{5}{2}\right) \xi_0 = 0. \tag{6.9}$$

This concludes the formal solution.

It is clear that a general discussion on the lines followed by Davis *et al.* (1976) is not possible here because of the complexity of the above relations, and so we restrict the remainder of this section to a discussion of the particular case where $\xi_0 = 1$. Other cases have been considered, but the present example appears to be sufficiently illustrative. The coefficients decay fairly rapidly, and $r_n$, $s_n$ can be neglected for $n > 8$, $t_n$ for $n > 6$, to give satisfactory accuracy. Separated flow is found to exist. The adjacent poles of the two spheres both attract fluid, in opposition to the general effect of the spheres, when considered as a single rotating

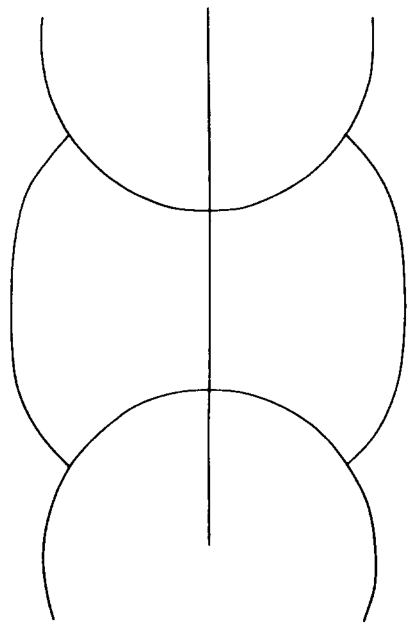

Figure 2. Separated region for the two equal spheres with $\xi_0 = 1$.

body, to spin fluid out along the equator $z = 0$; separated flow can be seen as a possible consequence of this simple kinematic understanding. The streamline $\psi = 0$ is shown in figure 2. There is fore–aft symmetry, and so no resultant translational force exists.

## 7. DISCUSSION

Ranger (1971) considered the combination of a uniform flow past a sphere that is itself rotating, and calculated the secondary streaming motion. When the velocity for the uniform flow is $U$, then the two effects will have equal magnitude when $U$ and $\Omega^2 a^3/\nu$ are of the same order, given that the Rossby number is finite. That is, if we write $R = Ua/\nu$ as the Reynolds number for the uniform flow, then it is necessary that $RE^2 = O(1)$ for a balance. The combination of the two basic flows is linear, and so it is sufficient just to add the two solutions together. Ranger found that the flow separated at the rear of the sphere. When the fluid at infinity is also rotating, there is an additional interesting feature. We combine [2.5] with the classic uniform flow solution

$$U\left(\frac{1}{2}r^2 - \frac{3}{4}r + \frac{1}{4}r^{-1}\right)\sin^2\theta,$$

for the stream function, and see that when

$$RE^2 < \frac{1}{2}(\lambda - 1)$$

the region of separation still extends as $r$ increases to infinity. Hence, the separated flow enters the "Oseen" domain with distances that are $O(E^{1/2})$ away from the sphere. Clearly this region is closed, but it requires calculations within the "Oseen" domain to effect the closure.

When the Rossby number is small, then it is necessary that $RE^2 = O(\epsilon)$ for a balance between the uniform stream and secondary streaming flows. For each of the different geometries considered here the corresponding uniform stream results have already been referenced in the literature. In the particular case of the finite disc, we combine [4.4] with the stream function

$$\frac{1}{2} U \sin^2 \eta[(\tau^2+1) - 2\pi^{-1}\{(\tau^2+1) \operatorname{arc\,cot} \tau + \tau\}]$$

for the uniform flow case.

It can be seen that separation takes place at the rim of the disc, and the angle the streamline $\psi = 0$ makes with the disc is

$$\arctan [48\pi\alpha(64\alpha^2 - 9\pi^2)^{-1}], \quad \alpha = RE^2/\epsilon \ ;$$

this angle takes on all values between 0 and $\pi$.

*Acknowledgements*—The author wishes to thank Keith Ranger for accelerating his understanding of many of the techniques used in solving the Stokes flow equations. The financial assistance of NSERC Grant A-4059 is also gratefully acknowledged.

REFERENCES

Childress, S. 1964 The slow motion of a sphere in a rotating, viscous fluid. *J. Fluid Mech.* **20**, 305–314.

Collins, W. D. 1959 On the solution of some axisymmetric boundary value problems by means of integral equations. *Q. J. Mech. Appl. Math.* **12**, 232–241.

Davis, A. M. J., O'Neill, M. E., Dorrepaal, J. M. & Ranger, K. B. 1976 Separation from the surface of two equal spheres in Stokes flow. *J. Fluid Mech.* **77**, 625–644.

Dorrepaal, J. M., O'Neill, M. E. & Ranger, K. B. 1976 Axisymmetric Stokes flow past a spherical cap. *J. Fluid Mech.* **75**, 273–286.

Hasimoto, H. & Sano, O. 1980 Stokeslets and eddies in creeping flow. *Ann. Rev. Fluid Mech.* **12**, 335–364.

Herron, I. H., Davis, S. H & Bretherton, F. P. 1975 On the sedimentation of a sphere in a centrifuge. *J. Fluid Mech.* **68**, 209–234.

O'Neill, M. E. & Ranger, K. B. 1979 Laminar particle-fluid interaction. In *Handbook of Multiphase Flow* (Edited by G. Hetsroni). Hemisphere, Washington.

Payne, L. E. & Pell, W. H. 1960 The Stokes flow problem for a class of axially symmetric bodies. *J. Fluid Mech.* **7**, 529–549.

Ranger, K. B. 1971 Slow viscous flow past a rotating sphere. *Proc. Camb. Phil. Soc.* **69**, 333–336.

Ranger, K. B. 1972 Integral transforms in axisymmetric potential theory. M. R. C. Technical Summary Rep. No. 1225, Univ. of Wisconsin.